Univalent Harmonic Mappings and Linearly Connected Domains

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Abstract

We investigate the relationship between the univalence of f and of h in the decomposition $f = h + \overline{g}$ of a sense-preserving harmonic mapping defined in the unit disk $\mathbb{D} \subset \mathbb{C}$. Among other results, we determine the holomorphic univalent maps h for which there exists $c > 0$ such that every harmonic mapping of the form $f = h + \overline{g}$ with $|g'| < c|h'|$ is univalent. The notion of a linearly connected domain appears in our study in a relevant way.

A planar harmonic mapping is a complex-valued harmonic function $f(z)$, $z = x + iy$, defined on some domain $\Omega \subset \mathbb{C}$. When Ω is simply connected, the mapping has a canonical decomposition $f = h + \overline{g}$, where h and g are analytic in Ω. Since the Jacobian of f is given by $|h'|^2 - |g'|^2$, it is locally univalent and orientation-preserving if and only $|g'| < |h'|$, or equivalently if $h'(z) \neq 0$ and the dilatation $\omega = g'/h'$ has the property $|\omega(z)| < 1$ in Ω.

Fundamental questions regarding univalent harmonic mappings are still to be resolved, including important coefficient estimates, and the exact nature of the analogue of the Riemann mapping theorem. There are beautiful results such as the shear construction of Clunie and Sheil-Small [C-SS], and the theorem of Radó-Kneser-Choquet for convex harmonic mappings [K], [Ch]. The literature nevertheless appears to contain few results about such basic issues as the relation between the univalence of f and of h. In this paper we determine conditions under which the univalence of one of them implies that of the other. We also find conditions under which the harmonic mappings $F = h + e^{i\theta}\overline{g}$ remain univalent for all $\theta \in [0, 2\pi]$.

A domain $\Omega \subset \mathbb{C}$ is *linearly connected* if there exists a constant $M < \infty$ such that any two points $w_1, w_2 \in \Omega$ are joined by a path $\gamma \subset \Omega$ of length $l(\gamma) \leq M|w_1 - w_2|$, or equivalently (see [P]), $\mathrm{diam}(\gamma) \leq M|w_1 - w_2|$. Such a domain is necessarily a Jordan domain, and for piecewise smoothly bounded domains, linear connectivity is equivalent to the boundary's having no inward-pointing cusps. Our first result (Theorem 1, below) can be considered a harmonic version of the main theorem in [A-B], included here for the convenience of the reader.

Theorem A: *Let $\Omega \subset \mathbb{C}$ be simply connected. Then Ω is linearly connected if and only if there exists $\epsilon > 0$ such that the condition $|\Phi'(w) - 1| < \epsilon$ for all $w \in \Omega$ implies the analytic map $\Phi : \Omega \to \mathbb{C}$ is injective.*

We now state:

Theorem 1: *Let $h : \mathbb{D} \to \mathbb{C}$ be a holomorphic univalent map. Then there exists $c > 0$ such that every harmonic mapping $f = h + \overline{g}$ with dilatation $|\omega| < c$ is univalent if and only if $h(\mathbb{D})$ is a linearly connected domain.*

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The proof of Theorem 1 will show that one can take $c = 1$ when h is *convex*, an important special case that we state separately as a corollary.

Corollary: *Let h be analytic and convex in $\mathbb{D}$. Then every harmonic mapping of the form $f = h + \overline{g}$ with $|\omega| < 1$ is injective.*

In the remarks after the proof of Theorem 1 it will be shown that one may take $c = 1$ only when h is a convex map.

Proof of Theorem 1: Suppose first that $\Omega = h(\mathbb{D})$ is linearly connected, and let $f = h + \overline{g}$ be a harmonic mapping with $|\omega| < 1/M$. We claim that f is univalent in $\mathbb{D}$, or equivalently, that $w + \overline{\varphi(w)}$ is univalent in Ω, where $\varphi = g \circ h^{-1}$ satisfies $|\varphi'| < 1/M$. If not, then for $w_1 \neq w_2$ we will have

$$\varphi(w_2) - \varphi(w_1) = \overline{w_1 - w_2}\,. \tag{1}$$

Let $\gamma \subset \Omega$ be a path joining w_1 and w_2 with $l(\gamma) \leq M|w_1 - w_2|$. Then

$$|\varphi(w_2) - \varphi(w_1)| \leq \int_\gamma |\varphi'(w)||dw| < \frac{1}{M}\int_\gamma |dw| \leq |w_1 - w_2|\,,$$

which contradicts (1). When Ω is convex then we may take $M = 1$, and thus f will be univalent as long as $|\omega| < 1$.

Suppose now that Ω is not linearly connected. We seek nonunivalent harmonic mappings of the form $f = h + \overline{g}$ with arbitrarily small dilatation. According to Theorem A, for every $c > 0$ there exists a nonunivalent holomorphic function $\Phi : \Omega \to \mathbb{C}$ that satisfies $|\Phi'(w) - 1| < c$ on Ω. We write $\Phi(w) = w + \varphi(w)$, so that $|\varphi'(w)| < c$. Let w_1, w_2 be distinct points for which $\Phi(w_1) = \Phi(w_2)$, that is, points for which $w_1 - w_2 = \varphi(w_2) - \varphi(w_1)$. Let θ be such that $e^{i\theta}(w_1 - w_2) \in \mathbb{R}$. We claim that the harmonic mapping $F(w) = w + e^{-2i\theta}\overline{\varphi(w)}$ fails to be univalent in Ω. Indeed,

$$w_1 - w_2 = e^{-i\theta}e^{i\theta}(w_1 - w_2) = e^{-i\theta}\overline{e^{i\theta}(w_1 - w_2)} = e^{-2i\theta}\overline{w_1 - w_2} = e^{-2i\theta}\overline{\varphi(w_2) - \varphi(w_1)}\,,$$

that is, $F(w_1) = F(w_2)$. The nonunivalent harmonic mapping $f(z) = F(h(z))$ has dilatation bounded by c. This finishes the proof.

We point out that in the case that $\Omega = h(\mathbb{D})$ is linearly connected with constant M and $|\omega| \leq c$ for some $c < 1/M$, then $R = f(\mathbb{D})$ is also linearly connected. In effect, let us regard R as the image of Ω under the harmonic mapping $F(w) = w + \overline{\varphi(w)}$, where $\varphi = g \circ h^{-1}$ satisfies $|\varphi'| \leq c$. Let $\zeta_1 = F(w_1)$, $\zeta_2 = F(w_2) \in R$ be distinct points. By assumption, there exists $\gamma \subset \Omega$ joining w_1 and w_2 such that $l(\gamma) \leq M|w_1 - w_2|$. Let $\Gamma = F(\gamma)$. Because the complex derivatives of F satisfy $|F_w| + |F_{\overline{w}}| \leq 1 + c$, it follows that $|l(\Gamma)| \leq (1 + c)l(\gamma) \leq (1 + c)M|w_1 - w_2|$. On the other hand,

$$\begin{aligned}|\zeta_1 - \zeta_2| &= |w_1 - w_2 + \overline{\varphi(w_1) - \varphi(w_2)}| \geq |w_1 - w_2| - |\varphi(w_1) - \varphi(w_2)| \\ &\geq |w_1 - w_2| - \int_\gamma |\varphi'(w)||dw| \geq |w_1 - w_2| - cl(\gamma) \geq (1 - cM)|w_1 - w_2|\,,\end{aligned}$$

and therefore

$$l(\Gamma) \leq \frac{(1 + c)M}{1 - cM}\,|\zeta_1 - \zeta_2|\,,$$

that is, R is linearly connected with constant $(1 + c)M/(1 - cM)$.

The case when c may be taken equal to 1 in Theorem 1 deserves special attention. Suppose h is a univalent map with the property that $h + \overline{g}$ is univalent for all g with $|g'| < |h'|$. We claim that

$\Omega = h(\mathbb{D})$ must be a convex domain. If not, it follows from [H-P] that there exists a nonunivalent holomorphic map $\Psi : \Omega \to \mathbb{C}$ with $\mathrm{Re}\{\Psi'(w)\} > 0$ for all $w \in \Omega$. It is easy to modify Ψ to a mapping Φ that remains nonunivalent in Ω with $\mathrm{Re}\{\Phi'(w)\}$ contained in a compact subset of the right half plane $\mathbb{H}$. For example, let $p(\zeta, t)$ be a family of univalent mappings, holomorphic in ζ and continuous in t, which map $\mathbb{H}$ onto proper subsets of $\mathbb{H}$, and which converge locally uniformly to $p(\zeta, 0) = \zeta$ as $t \to 0$. The antiderivatives of $p(\Psi'(w), t)$ appropriately chosen converge locally uniformly to Ψ as $t \to 0$, and we may therefore take Φ to be the antiderivative of $p(\Psi'(w), t)$ for small t. By scaling the range $\Phi'(\Omega)$ we may assume that it is contained in the disk $|\zeta - 1| < 1$, and we can write $\Phi(w) = w + \varphi(w)$, with $|\varphi'(w)| < 1$. A shown above, for suitable θ, the harmonic mapping $F(w) = w + e^{i\theta}\overline{\varphi(w)}$ will fail to be univalent in Ω, and hence $f(z) = h(z) + \overline{g(z)}$ with $g(z) = e^{-i\theta}\varphi(h(z))$ is a nonunivalent harmonic mapping with $|\omega| < 1$.

In our second result, the univalence of h is deduced from that of f.

Theorem 2: *Let $f = h + \overline{g}$ be a sense-preserving univalent harmonic mapping defined on $\mathbb{D}$, and suppose that $\Omega = f(\mathbb{D})$ is linearly connected with constant M. If $|\omega| < 1/(1+M)$ then h is univalent.*

Proof: Suppose that $h(z_1) = h(z_2)$ for distinct points $z_1, z_2 \in \mathbb{D}$. Then $f(z_1) - f(z_2) = \overline{g(z_1) - g(z_2)}$, that is,

$$\overline{w_1 - w_2} = G(w_1) - G(w_2)\,, \tag{2}$$

where $w = f(z)$ and $G = g \circ f^{-1}$. We will estimate the complex derivatives $G_w = g' \cdot (f^{-1})_w$, $G_{\overline{w}} = g' \cdot (f^{-1})_{\overline{w}}$ to show that (2) leads to a contradiction. Differentiation of the equation $f^{-1}(f(z)) = z$ yields the relations

$$(f^{-1})_w \cdot h' + (f^{-1})_{\overline{w}} \cdot g' = 1\,,$$

$$(f^{-1})_w \cdot \overline{g'} + (f^{-1})_{\overline{w}} \cdot \overline{h'} = 0\,,$$

hence

$$(f^{-1})_w = \frac{\overline{h'}}{|h'|^2 - |g'|^2} \quad , \qquad (f^{-1})_{\overline{w}} = -\frac{\overline{g'}}{|h'|^2 - |g'|^2}\,. \tag{3}$$

It follows that

$$|G_w| + |G_{\overline{w}}| = |g'|\frac{|h'| + |g'|}{|h'|^2 - |g'|^2} = \frac{|g'|}{|h'| - |g'|} = \frac{|\omega|}{1 - |\omega|} < \frac{1}{M} \tag{4}$$

because $|\omega| < 1/(1+M)$ by assumption.

Let $\gamma \subset \Omega$ be a curve joining w_1, w_2 with $l(\gamma) \le M|w_1 - w_2|$. Then

$$|G(w_1) - G(w_2)| \le \int_\gamma (|G_w| + |G_{\overline{w}}|)\,|dw| < \frac{l(\gamma)}{M} \le |w_1 - w_2|\,,$$

which stands in contradiction with (2). This finishes the proof.

As before, one can show that when the dilatation of $f = h + \overline{g}$ satisfies the stricter bound $|\omega| \le c$ for some $c < 1/(1+M)$, then $h(\mathbb{D})$ is a linearly connected domain. To this effect, we regard $R = h(\mathbb{D})$ as the image of $\Omega = f(\mathbb{D})$ under the mapping $H(w) = w - \overline{G(w)}$, where $G = g \circ f^{-1}$.

Let $\zeta_i = H(w_i) \in R$, $i = 1, 2$, and let $\gamma \subset \Omega$ be a curve joining w_1 and w_2 with $l(\gamma) \leq M|w_1 - w_2|$. We will show that the curve $\Gamma = H(\gamma)$ satisfies

$$l(\Gamma) \leq \frac{M}{1 - c(1 + M)} \, |\zeta_1 - \zeta_2| \, . \tag{5}$$

Using equations (3) we find that

$$H_w = 1 - \overline{g' \cdot (f^{-1})_{\overline{w}}} = 1 + \frac{|g'|^2}{|h'|^2 - |g'|^2} = \frac{|h'|^2}{|h'|^2 - |g'|^2} \, ,$$

$$H_{\overline{w}} = -\, \overline{g' \cdot (f^{-1})_w} = -\frac{h'\overline{g'}}{|h'|^2 - |g'|^2} \, ,$$

therefore

$$|H_w| + |H_{\overline{w}}| = \frac{|h'|}{|h'| - |g'|} = \frac{1}{1 - |\omega|} \leq \frac{1}{1 - c} \, .$$

We conclude that

$$l(\Gamma) \leq \int_\gamma (|H_w| + |H_{\overline{w}}|)|dw| \leq \frac{1}{1 - c} \, l(\gamma) \leq \frac{M}{1 - c} \, |w_1 - w_2| \, . \tag{6}$$

On the other hand, since $\zeta_1 - \zeta_2 = w_1 - w_2 - \overline{G(w_1) - G(w_2)}$ we have that

$$|\zeta_1 - \zeta_2| \geq |w_1 - w_2| - |G(w_1) - G(w_2)| \geq |w_1 - w_2| - \int_\gamma (|G_w| + |G_{\overline{w}}|)|dw|$$

$$= |w_1 - w_2| - \int_\gamma \frac{|\omega|}{1 - |\omega|} |dw| \geq |w_1 - w_2| - \frac{c}{1 - c} \, l(\gamma)$$

$$\geq |w_1 - w_2| - \frac{Mc}{1 - c} \, |w_1 - w_2| = \frac{1 - c(1 + M)}{1 - c} \, |w_1 - w_2| \, . \tag{7}$$

The inequality (5) follows from (6) and (7).

An interesting case of Theorem 2 is when $f = h + \overline{g}$ is a convex harmonic mapping, that is, when $\Omega = f(\mathbb{D})$ is linearly connected with constant 1. Then, according to Theorem 5.7 in [C-SS], the holomorphic part h will be univalent (in fact, close to convex) regardless of any restriction on the dilatation ω. Thus our theorem is far from optimal when $M = 1$. We do not know if it ever becomes optimal, especially, for large values of M.

In our last result, the notion of linearly connected domain is used to describe certain classes of univalent harmonic maps f, in a sense, stable.

Theorem 3: *Let $f = h + \overline{g}$ be a sense-preserving univalent harmonic mapping defined on $\mathbb{D}$, and suppose that $\Omega = f(\mathbb{D})$ is linearly connected with constant M. If $|\omega| < 1/(1 + 2M)$ then $F = h + e^{i\theta}\overline{g}$ is univalent for every θ.*

Proof: Let $F = h + e^{i\theta}\overline{g}$ and write $F = f + (e^{i\theta} - 1)\overline{g}$. If F fails to be univalent, then for some distinct points $z_1, z_2 \in \mathbb{D}$,

$$f(z_2) - f(z_1) = (1 - e^{i\theta})\overline{g(z_2) - g(z_1)} \, ,$$

that is,

$$\overline{w_2 - w_1} = (1 - e^{-i\theta})(G(w_2) - G(w_1)) \, , \tag{8}$$

where $w = f(z)$ and $G = g \circ f^{-1}$. As in (4) we have that

$$|G_w| + |G_{\overline{w}}| = \frac{|\omega|}{1-|\omega|} < \frac{1}{2M} \tag{9}$$

because $|\omega| < 1/(1+2M)$ by assumption.

Let $\gamma \subset \Omega$ be a curve joining w_1, w_2 with $l(\gamma) \leq M|w_1 - w_2|$. Then

$$|G(w_2) - G(w_1)| \leq \int_\gamma (|G_w| + |G_{\overline{w}}|)\,|dw| < \frac{l(\gamma)}{2M} \leq \frac{|w_2 - w_1|}{2}\,,$$

which contradicts (8). This finishes the proof.

Theorem 3 can also be deduced from Theorem 1 and the remarks to Theorem 2, to draw a stronger conclusion. In effect, let $f = h + \overline{g}$ satisfy the hypotheses of Theorem 3. Because $|\omega| < c = 1/(1+2M) < 1/(1+M)$, it follows from equation (5) that $h(\mathbb{D})$ is a linearly connected domain with constant

$$\frac{M}{1-c(1+M)} = \frac{M}{1-(1+M)/(1+2M)} = 1+2M\,.$$

Hence, we conclude from (the proof of) Theorem 1 that any harmonic mapping of the form $h + \overline{\varphi}$ will be univalent if $|\varphi'| < |h'|/(1+2M)$, which includes, in particular, all rotations $\varphi = e^{i\theta} g$.

Finally, an important instance of Theorem 3 is when $\Omega = f(\mathbb{D})$ is convex, in which case any harmonic mapping of the form $F = h + e^{i\theta}\overline{g}$ will be univalent if $|\omega| < 1/3$. We have been unable to prove or disprove that the constant $1/3$ is sharp.

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